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STABILITY FOR ABSTRACT LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

A class of parabolic partial integrodifferential equations with discrete and distributed delays in the spatial derivatives of maximum order is considered. After the study of well posedness of the initial value problem the asymptotic behaviour of the solutions is investigated through the spectral properties of the infinitesimal generator of the solution semigroup.

1. Introduction

The subject of retarded partial differential equations has been investigated by several authors in the last years but very few papers have been devoted to the study of the case in which the delays appear in the partial derivatives with respect to the space variables (see [1–4, 7, 8, 12, 15, 17, 18]). Our objective in this work is to investigate a class of parabolic linear partial integrodifferential equations with delays up to the highest order derivatives. The simplest example that could be considered is the following:

$$u_t(t,x) = u_{xx}(t,x) + u_{xx}(t-r,x) + \int_{-r}^{0} a(s)u_{xx}(t+s,x)ds + f(t,x), \tag{1.1}$$

$$\text{for } 0 < t \leqq T, \quad 0 \leqq x \leqq 1,$$

$$u(t,x) = \varphi(t,x), \qquad \text{for } -r \leqq t \leqq 0, \quad 0 \leqq x \leqq 1, \tag{1.2}$$

$$u(t,0) = u(t,1) = 0, \qquad \text{for } 0 < t \leqq T, \tag{1.3}$$

where f and φ are given functions and r is a positive real number. We can treat

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this problem as an initial value problem for an abstract ordinary functional differential equation of the form

$$(1.4)\qquad \dot{u}(t)=Au(t)+A_1u(t-r)+\int_{-r}^{0}a(s)A_2u(t+s)ds+f(t),\quad \text{for } 0<t\leqq T,$$

$$(1.5)\qquad u(t)=y(t),\qquad \text{for } -r\leqq t\leqq 0,$$

where $A : D_A \subseteq H \to H$ is the infinitesimal generator of an analytic semigroup in a Banach space H and $A_1, A_2 \in \mathscr{L}(D_A, H)$ where D_A is endowed with the graph norm of A. Let us briefly review the existing literature on this subject. Problem (1.4)–(1.5) has been studied in [17], where a mild solution is proved to exist, if y is suitably chosen. In the example (1.1)–(1.3), φ is supposed to be infinitely differentiable with respect to x. The abstract problem with $A_2=0$ is considered in [1]. Here the initial function y must be continuous with values in an interpolation space between D_{A^2} and D_A. This is sufficient to obtain a strict solution of (1.4) and (1.5).

In this paper we shall study (1.4)–(1.5) in a Hilbert space setting. This is useful, for instance, in approximation problems and in some applications to control theory (see [7]). In this case we will assume that H is a Hilbert space, we choose $y \in L^2(-r, 0; D_A)$, $f \in L^2(0, T; H)$ and look for a solution u such that Au, $\dot{u} \in L^2(0, T; H)$; hence (1.4)–(1.5) is satisfied almost everywhere. It is known that we must also supplement (1.5) with the prescription of the value of u at $t=0$:

$$(1.6)\qquad u(0)=x$$

and choose x in a suitable intermediate space F between D_A and H (see [7]). With this assumption the solution obtained belongs to $L^2(-r, T; D_A) \cap W^{1,2}(0, T; H) \cap C(0, T; F)$. A generalized solution of the same problem is found under somewhat weaker conditions on x, y and f in [13]. In [12] a case is studied in which $A_1=0$ but a distributed delay term in the first time derivative is added; here more regular y and u are considered, hence condition (1.6) is not needed.

Taking $f=0$ in (1.4) we can associate to each $(x, y) \in F \times L^2(-r, 0; D_A) = Z$ the solution of (1.4)–(1.6) which is defined now in $[-r, +\infty[$. This enables us to define the solution semigroup $S(t)$ in the product space Z. In this way the study of the stability of the solution u of (1.4)–(1.6) is made through the investigation of the asymptotic behaviour of

$$(1.7)\qquad t \to S(t)(x, y)$$

in Z, i.e. the determination of the type ω of the semigroup $S(.)$. To this end we characterize its infinitesimal generator Λ and try to locate its spectrum $\sigma(\Lambda)$. If the condition

$$\omega = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\Lambda)\} \tag{1.8}$$

holds we can reduce the study of the asymptotic behaviour of the solution of (1.4)–(1.6) to that of the location of $\sigma(\Lambda)$. Now condition (1.8) holds, for example, if $H = \mathbf{R}^n$ (see [11]) or if e^{At} is compact and A_1, A_2 are defined in domains which contain properly D_A (this corresponds to partial differential equations with delays in the lower order derivatives); in fact, in both situations one can prove that $S(t)$ is compact for some t. But if

$$A_1 \in \mathscr{L}(D_A, H) \quad \text{and} \quad D_A \neq H \tag{1.9}$$

we can show that $S(t)$ is never compact. Another sufficient condition to verify (1.8) is the differentiability of the semigroup $S(.)$; but not even this property holds if A_1 satisfies (1.9). On the other hand when $A_1 = 0$ and $a(.)$ is smooth, the semigroup $S(.)$ is differentiable and so (1.8) holds. This case provides an interesting example of a semigroup which is not compact but ultimately differentiable.

In the next section we give the existence and uniqueness results for the (global) solution of problem (1.4)–(1.6) together with the continuous dependence on the initial data (x, y). After defining the solution semigroup we characterize its infinitesimal generator Λ.

In Section 3 we study the spectrum of Λ through the investigation of a generalization of the classical characteristic equation associated with (1.4)–(1.6). Again the presence of the delay in the derivatives of maximum order gives rise to new and interesting situations.

In Section 4 we investigate the discrete delay case, which is problem (1.4)–(1.6) with $A_2 = 0$, and apply the results of the preceding section to the case in which $A_1 = \gamma A$ with $\gamma \in R$ and $|\gamma| > 1$. When $0 < |\gamma| \leq 1$ we use the methods of dissipative operators theory to determine the asymptotic behaviour of the solution semigroup.

In Section 5 we consider problem (1.4)–(1.6) when $A_1 = 0$, which is the distributed delay case. Supposing that the density function $a(.)$ belongs to $W^{1,2}(-r, 0)$ we can prove that the solution semigroup is differentiable in $]r, +\infty[$ and so we can study the asymptotic behaviour of the solutions of (1.4)–(1.6) by locating the spectrum of Λ. This will be done in detail for $A_2 = A$ and $a(.)$ a constant.

In the last section we give an application of the preceding results to the study of the classical Cauchy–Dirichlet problem for a parabolic second order integro-differential equation with delay.

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2. Wellposedness and solution semigroup

In this paper H will denote a Hilbert space with norm $\|.\|$. We will also consider a linear operator $A : D_A \subseteq H \to H$ which generates a bounded analytic semigroup e^{At}. The space D_A will always be given the norm of the graph of A. Let us denote by F the Hilbert space (see [5, 14])

$$F = \left\{ x \in H : \int_0^{+\infty} \|Ae^{At}x\|^2 dt < +\infty \right\}, \tag{2.1}$$

with the norm

$$\|x\|_F = \left(\|x\|^2 + \int_0^{+\infty} \|Ae^{At}x\|^2 dt \right)^{1/2}. \tag{2.2}$$

It is known that F is an intermediate space between D_A and H,

$$D_A \hookrightarrow F \hookrightarrow H, \tag{2.3}$$

and moreover for each $T > 0$ we have

$$L^2(0, T; D_A) \cap W^{1,2}(0, T; H) \hookrightarrow C(0, T; F). \tag{2.4}$$

Here $\hookrightarrow$ denotes algebraic and topological inclusion. This result can be found in ([14], Theorem 3.1, p. 23; see also the Appendix of [8]).

We will give now an existence and uniqueness result for a solution of problem (1.4)–(1.6) and also establish continuous dependence on the initial data. These properties will enable us to define a strongly continuous solution semigroup.

THEOREM 2.1. *Let* $A_i \in \mathcal{L}(D_A, H)$ $(i = 1, 2)$, $a \in L^2(-r, 0)$ *with* $r > 0$, $f \in L^2(0, T; H)$ *with* $T > 0$, $x \in F$ *and* $y \in L^2(-r, 0; D_A)$. *There exists a unique* $u \in L^2(-r, T; D_A) \cap W^{1,2}(0, T; H)$ *such that*

$$\dot{u}(t) = Au(t) + A_1 u(t - r) + \int_{-r}^{0} a(s) A_2 u(t + s) ds + f(t), \tag{2.5}$$

for a.e. $t \in [0, T]$,

(2.6) $$u(0)=x, \quad u(t)=y(t), \qquad \text{for a.e. } t\in[-r,0].$$

Moreover $u\in C(0,T;F)$ *and there is* $C>0$ *independent of* f, x *and* y *such that*

(2.7) $$\|u\|_{L^2(0,T;D_A)\cap W^{1,2}(0,T;H)}\leqq C\{\|f\|_{L^2(0,T;H)}+\|x\|_F+\|y\|_{L^2(-r,0;D_A)}\}.$$

The result can be obtained easily from the proof of Theorem 3.3 of [8] where (2.5)-(2.6) is considered with $f=0$ (but otherwise under slightly more general assumptions). The proof is based on a maximal regularity result for the solution of the equation

(2.8) $$\begin{aligned}&\dot u(t)=Au(t)+f(t), \qquad \text{for } 0<t\leqq T,\\ &u(0)=x,\end{aligned}$$

where $f\in L^2(0,T;H)$. To obtain a solution u of (2.5)-(2.6), which has no less regularity than the initial datum y, we need to take x in F. This is the reason for employing the space F here. If we choose $f=0$ in (2.5) we can associate with the couple (x,y) a solution u of (2.5)-(2.6) defined on $[-r,\infty[$ and therefore we can define a solution semigroup. To make this precise we require additional notation. Let

(2.9) $$Z=F\times L^2(-r,0;D_A)=\{(x,y)\colon x\in F,\ y\in L^2(-r,0;D_A)\}$$

with the norm

(2.10) $$\|(x,y)\|_Z=\|x\|_F+\|y\|_{L^2(-r,0;D_A)},$$

and let us suppose that the assumptions of Theorem 2.1 hold.

DEFINITION 2.2. Given $z=(x,y)\in Z$ let

$$u\in L^2_{\mathrm{loc}}(0,+\infty;D_A)\cap W^{1,2}_{\mathrm{loc}}(0,+\infty;H)\cap C(0,+\infty;F)$$

be the solution of (2.5)-(2.6) with $f=0$. For each $t>0$ we define the linear operator

(2.11) $$S(t)\colon Z\to Z$$

by

(2.12) $$S(t)(x,y)=(u(t),u_t)$$

where u_t is the function $s\to u(t+s)$, $s\in[-r,0]$. We put also $S(0)=I$.

The following result is proved in ([8], Theorem 4.1 and 4.2):

THEOREM 2.3. *The family of operators $S(.)$ is a strongly continuous semigroup on Z and its infinitesimal generator*

$$\Lambda: D_\Lambda \subseteq Z \to Z \tag{2.13}$$

is defined by

$$\begin{aligned} D_\Lambda = \Big\{(y(0), y)\colon y \in W^{1,2}(-r,0;D_A), Ay(0)+A_1y(-r) \\ +\int_{-r}^{0} a(s)A_2y(s)ds \in F\Big\}, \end{aligned} \tag{2.14}$$

and

$$\Lambda(y(0),y) = \left(Ay(0)+A_1y(-r)+\int_{-r}^{0} a(s)A_2y(s)ds, \dot{y}\right). \tag{2.15}$$

3. The characteristic equation

In this section we show how the description of the spectrum of Λ can be facilitated by introducing a family of operators $\Delta(\lambda)$, $\lambda \in \mathbf{C}$. The equation $\Delta(\lambda)x = 0$ will be seen to be a generalization of the well-known characteristic equation for delay equations when $H = R^n$ (see Chapter 7 of [11]) or when H is infinite dimensional but A_1 and A_2 are defined on the domain of a fractional power of $-A$ (see [18]). In the present case new situations arise.

The assumptions of Theorem 2.1 are assumed to hold throughout the remainder of this paper.

DEFINITION 3.1. For $\lambda \in \mathbf{C}$ we define $\Delta(\lambda)\colon D_A \to H$ by

$$\Delta(\lambda)x = \lambda x - Ax - e^{-\lambda r}A_1x - \left(\int_{-r}^{0} a(s)e^{\lambda s}ds\right) A_2x. \tag{3.1}$$

As D_A is given the norm of the graph of A, we have $\Delta(\lambda) \in \mathcal{L}(D_A, H)$.

We shall frequently use the following fact: If $\overline{\Delta(\lambda)D_A} = H$ and $\Delta(\lambda)^{-1}$ exists as a bounded operator, then $\Delta(\lambda)D_A = H$.

DEFINITION 3.2. For $\lambda \in \mathbf{C}$ we define $M_\lambda : L^2(-r,0;D_A) \to W^{1,2}(-r,0;D_A)$ by

$$(M_\lambda \bar{y})(s) = \int_{s}^{0} e^{\lambda(s-\tau)}\bar{y}(\tau)d\tau. \tag{3.2}$$

Further we define $G_\lambda : L^2(-r, 0; D_A) \to H$ by

$$G_\lambda(\bar{y}) = A_1(M_\lambda \bar{y})(-r) + \int_{-r}^{0} a(s) A_2(M_\lambda \bar{y})(s) ds. \tag{3.3}$$

It is easy to see that $G_\lambda \in \mathscr{L}(L^2(-r, 0; D_A), H)$. Its norm will be denoted by $\|G_\lambda\|$.

Proposition 3.3. *Let $\lambda \in C$ and $\bar{z} = (\bar{x}, \bar{y}) \in Z$. If $z = (y(0), y) \in D_\Lambda$ satisfies*

$$\lambda z - \Lambda z = \bar{z}, \tag{3.4}$$

then we have

$$y(t) = e^{\lambda t} y(0) + (M_\lambda \bar{y})(t), \qquad -r \leqq t \leqq 0, \tag{3.5}$$

and, setting $x = y(0)$, also

$$\Delta(\lambda) x = G_\lambda(\bar{y}) + \bar{x}. \tag{3.6}$$

Conversely, if $x \in D_A$ satisfies equation (3.6), *then for*

$$y(t) = e^{\lambda t} x + (M_\lambda \bar{y})(t), \qquad -r \leqq t \leqq 0, \tag{3.7}$$

we have that $y \in W^{1,2}(-r, 0; D_A)$, $z = (y(0), y) \in D_\Lambda$ and z satisfies (3.4).

Proof. Equation (3.4) can be written in the following way:

$$\lambda y(0) - A y(0) - A_1 y(-r) - \int_{-r}^{0} a(s) A_2 y(s) ds = \bar{x}, \tag{3.8}$$

$$\lambda y(t) - \dot{y}(t) = \bar{y}(t), \qquad \text{for a.e. } t \in [-r, 0], \tag{3.9}$$

and (3.9) is equivalent to (3.5). Hence if (3.4) holds we deduce that $x = y(0) \in D_A$ and that (3.6) is satisfied by virtue of (3.8) and (3.5). Conversely, if $x \in D_A$ then y defined by (3.7) belongs to $W^{1,2}(-r, 0; D_A)$ and $y(0) = x$. If in addition x satisfies (3.6) then from (3.7) we get

$$\begin{aligned}
&\lambda y(0) - A y(0) - A_1 y(-r) - \int_{-r}^{0} a(s) A_2 y(s) ds \\
&\quad = \lambda x - A x - A_1(e^{-\lambda r} x + (M_\lambda \bar{y})(-r)) - \int_{-r}^{0} a(s) A_2 \left(e^{\lambda s} x + \int_{s}^{0} e^{-\lambda \sigma} \bar{y}(\sigma) d\sigma \right) ds \\
&\quad = \bar{x}
\end{aligned}$$

and so (3.8) is verified. From (3.8) we deduce also

$$Ay(0)+A_1y(-r)+\int_{-r}^{0} a(s)A_2y(s)ds \in F$$

hence $(y(0),y)\in D_\Lambda$. As (3.8) and (3.9) hold we obtain (3.4).

By virtue of the preceding proposition we can study the relation between the properties of $\Delta(\lambda)$ and those of $\lambda-\Lambda$. This will be done in the following propositions.

PROPOSITION 3.4. *$\Delta(\lambda)$ is injective if and only if $\lambda-\Lambda$ is injective.*

PROOF. If $x\in D_A$, $x\neq 0$ and $\Delta(\lambda)x=0$ then x satisfies (3.6) with $(\bar{x},\bar{y})=(0,0)$ and so from Proposition 3.3 we deduce the existence of $z\in D_\Lambda$, $z\neq 0$ such that $\lambda z-\Lambda z=0$. Conversely, if there exists such a z, then from (3.6) we obtain $\Delta(\lambda)x=0$ with $x\in D_A$ and $x\neq 0$ and the conclusion is obtained.

PROPOSITION 3.5. *If $\Delta(\lambda)D_A=H$, then $(\lambda-\Delta)D_\Lambda=Z$.*

PROOF. If $\Delta(\lambda)D_A=H$ then (3.6) has a solution for each $(\bar{x},\bar{y})\in Z$ and so by virtue of Proposition 3.3 also (3.4) has a solution.

PROPOSITION 3.6. *If for $\lambda\in\mathbf{C}$ there exists $c_1>0$ such that*

$$\|x\|_{D_A}\leqq c_1\|\Delta(\lambda)x\| \tag{3.10}$$

for each $x\in D_A$, then there exists a constant c_2 such that

$$\|z\|_Z\leqq c_2\|(\lambda-\Lambda)z\|_Z \tag{3.11}$$

for each $z\in D_\Lambda$.

PROOF. Let us first observe that for $x\in D_A$ and $\bar{y}\in L^2(-r,0;D_A)$ the function

$$y(s)=e^{\lambda s}x+(M_\lambda\bar{y})(s),\qquad -r\leqq s\leqq 0, \tag{3.12}$$

satisfies

$$\|y\|_{L^2(-r,0;D_A)}\leqq c_3\{\|x\|_{D_A}+\|\bar{y}\|_{L^2(-r,0;D_A)}\}, \tag{3.13}$$

with c_3 independent of x and $\bar{y}$.

For $(y(0),y)=z\in D_\Lambda$ we set $\bar{z}=\lambda z-\Lambda z$ and $x=y(0)$. As (3.6) holds, (3.10) implies

$$\|x\|_{D_A}\leqq c_1\|\Delta(\lambda)x\|=c_1\|G_\lambda(\bar{y})+\bar{x}\|\leqq c_1\{\|G_\lambda\|\,\|\bar{y}\|_{L^2(-r,0;D_A)}+\|\bar{x}\|\}. \tag{3.14}$$

Next, observe that there exists a constant c_4 such that for each $x\in D_A$

$$\|x\|_F \leqq c_4 \|x\|_{D_A}$$

and hence from (3.13) and (3.14) it follows that

$$\begin{aligned}\|z\|_Z &= \|x\|_F + \|y\|_{L^2(-r,0;D_A)} \\ &\leqq c_4\|x\|_{D_A} + c_3\{\|x\|_{D_A} + \|\bar{y}\|_{L^2(-r,0;D_A)}\} \\ &\leqq (c_4 + c_3)c_1\{\|G_\lambda\| \|\bar{y}\|_{L^2(-r,0;D_A)} + \|\bar{x}\|\} + c_3\|\bar{y}\|_{L^2(-r,0;D_A)}.\end{aligned}$$

As $F \hookrightarrow H$ we deduce the existence of c_2 independent of $\bar{y}$ and $\bar{x}$ and such that

$$\|z\|_Z \leqq c_2\{\|\bar{y}\|_{L^2(-r,0;D_A)} + \|\bar{x}\|_F\} = c_2\|\bar{z}\|_Z = c_2\|\lambda z - \Lambda z\|_Z.$$

PROPOSITION 3.7. *If $(\lambda - \Lambda)D_\Lambda$ is dense in Z, then $\Delta(\lambda)D_A$ is dense in H.*

PROOF. Given $\bar{x} \in F$ set $\bar{z} = (\bar{x}, 0)$. By assumption for each $\varepsilon > 0$ there is $\bar{z}_\varepsilon = (\bar{x}_\varepsilon, \bar{y}_\varepsilon) \in Z$ such that

$$\|\bar{z}_\varepsilon - \bar{z}\|_Z < \varepsilon \tag{3.15}$$

and also such that there is $w_\varepsilon = (y_\varepsilon(0), y_\varepsilon) \in D_\Lambda$ satisfying

$$\lambda w_\varepsilon - \Lambda w_\varepsilon = \bar{z}_\varepsilon.$$

From Proposition 3.3 we deduce that

$$\Delta(\lambda) y_\varepsilon(0) = G_\lambda(\bar{y}_\varepsilon) + \bar{x}_\varepsilon. \tag{3.16}$$

Now

$$\|\bar{x} - G_\lambda(\bar{y}_\varepsilon) - \bar{x}_\varepsilon\| \leqq \|\bar{x} - \bar{x}_\varepsilon\| + \|G_\lambda(\bar{y}_\varepsilon)\|$$

and from (3.15) (and the fact that $F \hookrightarrow H$) we have

$$\|\bar{x} - \bar{x}_\varepsilon\| \leqq c_5\|\bar{x} - \bar{x}_\varepsilon\|_F < c_5\varepsilon$$

where $c_5 > 0$ is independent of $\bar{x}$ and $\bar{x}_\varepsilon$ and again from (3.15)

$$\|\bar{y}_\varepsilon\|_{L^2(-r,0;D_A)} < \varepsilon$$

hence

$$\|G_\lambda(\bar{y}_\varepsilon)\| \leqq \|G_\lambda\| \|\bar{y}_\varepsilon\|_{L^2(-r,0;D_A)} \leqq \|G_\lambda\|\varepsilon.$$

In conclusion, we proved that for $\bar{x} \in F$ and $\varepsilon > 0$ there exists $\xi_\varepsilon = G_\lambda(\bar{y}_\varepsilon) + \bar{x}_\varepsilon$ in the range of $\Delta(\lambda)$ such that $\|\bar{x} - \xi_\varepsilon\| \leqq \varepsilon(c_5 + \|G\|)$. Since F is dense in H the proposition is verified.

In analogy to the nomenclature used for the subdivision of the spectrum of closed operators we give the following

DEFINITION 3.8.

$$\rho(\Delta) = \{\lambda \in \mathbf{C}\colon \Delta(\lambda) \text{ is injective, } \Delta(\lambda)D_A = H,\ \Delta(\lambda)^{-1} \in \mathscr{L}(H, D_A)\},$$

$$\sigma_C(\Delta) = \{\lambda \in \mathbf{C}\colon \Delta(\lambda) \text{ is injective, } \overline{\Delta(\lambda)D_A} = H,\ \Delta(\lambda)^{-1} \text{ is unbounded}\},$$

$$\sigma_R(\Delta) = \{\lambda \in \mathbf{C}\colon \Delta(\lambda) \text{ is injective, } \overline{\Delta(\lambda)D_A} \neq H\},$$

$$\sigma_P(\Delta) = \{\lambda \in \mathbf{C}\colon \Delta(\lambda) \text{ is not injective}\}.$$

From the remark after Definition 3.1 we deduce that

$$\mathbf{C} = \rho(\Delta) \dot{\cup} \sigma_C(\Delta) \dot{\cup} \sigma_R(\Delta) \dot{\cup} \sigma_P(\Delta).$$

Here $E \dot{\cup} F$ denotes the disjoint union of two sets E and F.

By substituting in the preceding definitions $\lambda - \Lambda$ for $\Delta(\lambda)$, D_Λ for D_A and Z for H we obtain the usual definitions of $\rho(\Lambda)$, the resolvent of Λ, and of the continuous, residual and point spectrum of Λ: $\sigma_C(\Lambda)$, $\sigma_R(\Lambda)$ and $\sigma_P(\Lambda)$. We also denote by $\sigma(\Delta)$ and $\sigma(\Lambda)$ the complement in $\mathbf{C}$ of $\rho(\Delta)$ and $\rho(\Lambda)$ respectively.

Next we establish the result announced earlier on the relationship between the subsets of the spectrum of Λ and the sets defined in Definition 3.8.

THEOREM 3.9. *The following inclusions and equalities hold*:

$$\rho(\Delta) \subseteq \rho(\Lambda) \subseteq \rho(\Delta) \dot{\cup} \sigma_C(\Delta), \tag{3.17}$$

$$\sigma_C(\Lambda) \subseteq \sigma_C(\Delta) \subseteq \sigma_C(\Lambda) \dot{\cup} \sigma_R(\Lambda) \dot{\cup} \rho(\Lambda), \tag{3.18}$$

$$\sigma_R(\Delta) \subseteq \sigma_R(\Lambda) \subseteq \sigma_R(\Delta) \dot{\cup} \sigma_C(\Delta), \tag{3.19}$$

$$\sigma_P(\Lambda) = \sigma_P(\Delta), \tag{3.20}$$

$$\sigma(\Lambda) \subseteq \sigma(\Delta). \tag{3.21}$$

PROOF. The first inclusion in (3.17) is a consequence of Propositions 3.4, 3.5 and 3.6. The second one follows from Propositions 3.4 and 3.7. The first inclusion in (3.18) is true by virtue of Propositions 3.4, 3.6 and 3.7 and the other one is a consequence of Proposition 3.4. Moreover Propositions 3.4 and 3.7 imply the left hand side of (3.19), while the right hand side is true by virtue of Propositions 3.4 and 3.5. Finally (3.20) is equivalent to Proposition 3.4 and (3.21) is a consequence of (3.18)–(3.20).

In what follows we derive a condition under which also the converse of Proposition 3.6 is true. Yet another operator needs to be introduced.

DEFINITION 3.10. For each $\lambda \in \mathbf{C}$, the operator $\tilde{G}_\lambda : D_A \to H$ is given by

$$\tilde{G}_\lambda(x) = A_1(M_\lambda x_c)(-r) + \int_{-r}^{0} a(s) A_2(M_\lambda x_c)(s)\,ds, \tag{3.22}$$

where x_c denotes the constant function on $[-r, 0]$ with value x. We observe that $\tilde{G}_\lambda \in \mathscr{L}(D_A, H)$.

The converse of Proposition 3.6 holds if $\tilde{G}_\lambda - \mu$ has a bounded inverse on H for some $\mu \in \mathbf{C}$. Later we will see a situation where this property is easily verified.

PROPOSITION 3.11. *If there exist $\lambda, \mu \in \mathbf{C}$ and $\tilde{c}_1 > 0$ such that*

$$(\tilde{G}_\lambda - \mu)^{-1} \in \mathscr{L}(H, D_A) \tag{3.23}$$

and

$$\| z \|_Z \leqq \tilde{c}_1 \| (\lambda - \Lambda) z \|_Z \tag{3.24}$$

for each $z \in D_\Lambda$, then there exists $\tilde{c}_2 > 0$ satisfying

$$\| x \|_{D_A} \leqq \tilde{c}_2 \| \Delta(\lambda) x \| \tag{3.25}$$

for each $x \in D_A$.

PROOF. Suppose that (3.23) and (3.24) hold. From (3.24) we know that $\lambda - \Lambda$ is injective and by Proposition 3.4 the same is true for $\Delta(\lambda)$. Proceeding by contradiction we assume that there exist $\xi_n \in H$, $n \in \mathbf{N}$, with $\lim_{n\to\infty} \| \xi_n \| = 0$ and $\lim_{n\to\infty} \Delta(\lambda)^{-1} \xi_n \neq 0$ in D_A. For each $n \in \mathbf{N}$ we set

$$\bar{x}_n = -\mu(\tilde{G}_\lambda - \mu)^{-1} \xi_n$$

and

$$y_n(t) = (\tilde{G}_\lambda - \mu)^{-1} \xi_n$$

for each $t \in [-r, 0]$. Then $\bar{z}_n = (\bar{x}_n, \bar{y}_n) \in Z$ and

$$\lim_{n\to\infty} \| \bar{z}_n \|_Z = 0. \tag{3.26}$$

Moreover $G_\lambda(\bar{y}_n) + \bar{x}_n = \tilde{G}_\lambda(\tilde{G}_\lambda - \mu)^{-1}\xi_n - \mu(\tilde{G}_\lambda - \mu)^{-1}\xi_n = \xi_n = \Delta(\lambda)\Delta(\lambda)^{-1}\xi_n$. From (3.6) and the second part of Proposition 3.3 we conclude that

$$\lambda z_n - \Lambda z_n = \bar{z}_n, \tag{3.27}$$

where $z_n = (x_n, y_n)$ with

$$x_n = \Delta(\lambda)^{-1}\xi_n \tag{3.28}$$

and

$$y_n(t) = e^{\lambda t}x_n + (M_\lambda \bar{y}_n)(t) \tag{3.29}$$

for $t \in [-r, 0]$. If we prove that $\lim_{n\to\infty} z_n \neq 0$ in Z, then by (3.26) and (3.27) we obtain a contradiction to (3.24). Convergence of z_n to 0 in Z, however, implies convergence of y_n to 0 in $L^2(-r, 0; D_A)$. But by (3.28)–(3.29) we obtain for $t \in [-r, 0]$

$$y_n(t) = e^{\lambda t}\Delta(\lambda)^{-1}\xi_n + (\tilde{G}_\lambda - \mu)^{-1}\xi_n e^{\lambda t}\int_t^0 e^{-\lambda s}ds,$$

and consequently $\lim_{n\to\infty}\Delta(\lambda)^{-1}\xi_n = 0$ in D_A. This establishes the desired contradiction.

THEOREM 3.12. *If for each $\lambda \in \rho(\Lambda)$ there exists $\mu \in \mathbf{C}$ such that $(\tilde{G}_\lambda - \mu)^{-1} \in \mathscr{L}(H, D_A)$ then*

$$\rho(\Lambda) = \rho(\Delta), \tag{3.30}$$

$$\sigma_C(\Lambda) \subseteq \sigma_C(\Delta) \subseteq \sigma_C(\Lambda) \dot{\cup} \sigma_R(\Lambda), \tag{3.31}$$

$$\sigma_R(\Delta) \subseteq \sigma_R(\Lambda) \subseteq \sigma_R(\Delta) \dot{\cup} \sigma_C(\Delta), \tag{3.32}$$

$$\sigma_P(\Lambda) = \sigma_P(\Delta). \tag{3.33}$$

PROOF. If $\lambda \in \rho(\Lambda)$, then $\Delta(\lambda)$ is injective and $\Delta(\lambda)D_A$ is dense by virtue of Propositions 3.4 and 3.7. From Proposition 3.11 we deduce that $\Delta(\lambda)^{-1} \in \mathscr{L}(H, D_A)$, hence $\lambda \in \rho(\Delta)$ and $\rho(\Lambda) \subseteq \rho(\Delta)$. This together with (3.17) implies (3.30), which in turn implies (3.31) by virtue of (3.18). Finally (3.32) and (3.33) are (3.19) and (3.20).

By using the second part of Proposition 3.3 it is easy to demonstrate the next result:

PROPOSITION 3.13. *Let the following property hold: There exist subsets $F_1 \subseteq F$ and $L_1 \subseteq L^2(-r, 0; D_A)$, such that*

$$\overline{F_1 \times L_1} = Z \tag{3.34}$$

and $\lambda \in \mathbf{C}$ *such that*

$$F_1 + G_\lambda(L_1) \subseteq \Delta(\lambda) D_A. \tag{3.35$_\lambda$}$$

Then $(\lambda - \Lambda)D_\Lambda$ *is dense in* Z.

PROOF. Let $(\theta, \psi) \in F_1 \times L_1$. By assumption there exists $x \in D_A$ such that $\theta + G_\lambda \psi = \Delta(\lambda)x$. We define $\varphi(s) = e^{\lambda s}x + (M_\lambda \psi)(s)$ for $s \in [-r, 0]$ and note that $\varphi(0) = x$ and $\varphi' \in L^2(-r, 0; D_A)$. Moreover we find

$$\lambda\varphi(0) - \theta = A\varphi(0) + e^{-\lambda r}A_1\varphi(0) + \int_{-r}^{0} a(s)e^{\lambda s}ds A_2\varphi(0) + G_\lambda\psi$$

$$= A\varphi(0) + A_1\varphi(-r) + \int_{-r}^{0} a(s)A_2\varphi(s)ds.$$

Since $\lambda\varphi(0) - \theta \in F$ the last equality implies that

$$A\varphi(0) + A_1\varphi(-r) + \int_{-r}^{0} a(s)A_2\varphi(s)ds \in F,$$

and therefore $(\varphi(0), \varphi) \in D_\Lambda$.

We have shown that for each $(\theta, \psi) \in F_1 \times L_1$, there exists a preimage under the operator $\lambda - \Lambda$ in D_Λ. Since $F_1 \times L_1$ is dense in Z the claim is proved.

Finally we give conditions which allow us to completely describe the spectrum of Λ by means of $\Delta(\lambda)$:

THEOREM 3.14. *Suppose that* (3.34) *holds and for each* $\lambda \in \sigma_C(\Delta)$ *we have* (3.35)$_\lambda$ *and there exists* $\mu = \mu(\lambda)$ *such that* $(\tilde{G}_\lambda - \mu)^{-1} \in \mathscr{L}(H, D_A)$. *Then*

$$\rho(\Lambda) = \rho(\Delta), \tag{3.36}$$

$$\sigma_C(\Lambda) = \sigma_C(\Delta), \tag{3.37}$$

$$\sigma_R(\Lambda) = \sigma_R(\Delta), \tag{3.38}$$

$$\sigma_P(\Lambda) = \sigma_P(\Delta). \tag{3.39}$$

PROOF. If $\lambda \in \sigma_C(\Delta)$ we deduce from Proposition 3.13 that $(\lambda - \Lambda)D_\Lambda$ is dense in Z and from Proposition 3.4 that $\lambda - \Lambda$ is one-to-one; if $(\lambda - \Lambda)^{-1}$ is continuous then by virtue of Proposition 3.11 we have that $\Delta(\lambda)^{-1}$ is continuous which contradicts $\lambda \in \sigma_C(\Delta)$ and so $\lambda \in \sigma_C(\Lambda)$. Thus from (3.18) we obtain (3.37). From this and (3.17) we get (3.36) while (3.19) and (3.37) imply (3.38). Finally (3.39) is (3.33).

REMARK 3.15. Although we treat here only the case of one discrete delay, more general equations can be studied with the same techniques. For example let

$$L\varphi = \sum_{i=1}^{k} A_i\varphi(-r_i) + \int_{-r}^{0} a(s)A_{k+1}\varphi(s)ds,$$

when $A_i \in \mathscr{L}(D_A, H)$, $a \in L^2(-r,0)$ and $0 < r_1 < \cdots < r_k \leqq r$. Then generalizations of our results to the equation

$$\dot{u} = Au(t) + Lu_t$$

$$u(0) = x, \quad u(t) = y(t), \qquad \text{for a.e. } t \in [-r,0]$$

are simple.

4. The discrete delay case

We pass now to examine (2.5)–(2.6), when in addition to $f = 0$ we have also

$$A_2 = 0 \quad \text{and } A_1 = \gamma A \qquad \text{with } \gamma \in R - \{0\}, \tag{4.1}$$

so that the equation that we study is given by

$$\dot{u}(t) = Au(t) + \gamma Au(t-r), \qquad \text{for a.e. } t \geqq 0, \tag{4.2}$$

$$u(0) = x, \quad u(t) = y(t), \qquad \text{for a.e. } t \in [-r,0]. \tag{4.3}$$

Furthermore we suppose that $A \notin \mathscr{L}(H)$ so that

$$D_A \neq H. \tag{4.4}$$

For the present case Definition 3.1 gives for $\lambda \in \mathbf{C}$ and $x \in D_A$

$$\Delta(\lambda)x = \lambda x - m(\lambda)Ax \tag{4.5}$$

where

$$m(\lambda) = 1 + \gamma e^{-\lambda r}. \tag{4.6}$$

It is easy to check that

$$m(\lambda) = 0 \qquad \text{if and only if } \lambda \in M, \tag{4.7}$$

where

$$M = \left\{\frac{\log|\gamma|}{r} + i\frac{2\pi}{r}k + \lambda^* : k \in Z\right\} \quad \text{with } \lambda^* = \frac{i\pi}{r} \text{ if } \gamma > 0 \text{ and } \lambda^* = 0 \text{ if } \gamma < 0. \tag{4.8}$$

To use the general inclusions of Theorem 3.9 in our case, we need a preliminary result.

LEMMA 4.1. *Let*

$$\Gamma = \{\lambda \in \mathbf{C}: m(\lambda) \neq 0,\ \lambda m(\lambda)^{-1} \in \rho(A)\},$$

$$\Gamma_C = \{\lambda \in \mathbf{C}: m(\lambda) \neq 0,\ \lambda m(\lambda)^{-1} \in \sigma_C(A)\},$$

$$\Gamma_R = \{\lambda \in \mathbf{C}: m(\lambda) \neq 0,\ \lambda m(\lambda)^{-1} \in \sigma_R(A)\},$$

$$\Gamma_P = \{\lambda \in \mathbf{C}: m(\lambda) \neq 0,\ \lambda m(\lambda)^{-1} \in \sigma_P(A)\},$$

$$\Gamma_0 = \{\lambda \in \mathbf{C}: \lambda \neq 0,\ m(\lambda) = 0\},$$

then we have

$$\rho(\Delta) = \Gamma, \tag{4.9}$$

$$\sigma_C(\Delta) = \Gamma_C \dot{\cup} \Gamma_0, \tag{4.10}$$

$$\sigma_R(\Delta) = \Gamma_R, \tag{4.11}$$

$$\sigma_P(\Delta) = \begin{cases} \Gamma_P & \text{if } \gamma \neq -1, \quad (4.12) \\ \Gamma_P \dot{\cup} \{0\} & \text{if } \gamma = -1. \quad (4.13) \end{cases}$$

PROOF. If $m(\lambda) \neq 0$, then for each $x \in D_A$ we have

$$\Delta(\lambda)x = m(\lambda)(\lambda m(\lambda)^{-1} - A)x.$$

From this it easily follow that

$$\Gamma \subseteq \rho(\Delta), \quad \Gamma_C \subseteq \sigma_C(\Delta), \quad \Gamma_R = \sigma_R(\Delta), \quad \Gamma_P \subseteq \sigma_P(\Delta). \tag{4.14}$$

If $m(\lambda) = 0$ and $\lambda \neq 0$, then for each $x \in D_A$ we have $\Delta(\lambda)x = \lambda x$. In this case we deduce from (4.4) since $\lambda \neq 0$ that $\lambda \in \sigma_C(\Delta)$, which implies

$$\Gamma_0 \subseteq \sigma_C(\Delta). \tag{4.15}$$

In the case $m(0) \neq 0$ (i.e. $\gamma \neq -1$) we have

$$\mathbf{C} = \Gamma \dot{\cup} \Gamma_C \dot{\cup} \Gamma_R \dot{\cup} \Gamma_P \dot{\cup} \Gamma_0$$

and therefore (4.14)–(4.15) imply (4.9)–(4.12). On the other hand if $m(0) = 0$, we see that $\Delta(0) = 0$ and hence $0 \in \sigma_P(\Delta)$ and (4.12) must be replaced by (4.13).

PROPOSITION 4.2. *The following inclusions hold*:

$$\Gamma \subseteq \rho(\Lambda) \subseteq \Gamma \dot{\cup} \Gamma_C, \tag{4.16}$$

$$\Gamma_0 \subseteq \sigma_C(\Lambda) \subseteq \Gamma_0 \dot{\cup} \Gamma_C, \tag{4.17}$$

$$\Gamma_R \subseteq \sigma_R(\Lambda) \subseteq \Gamma_R \dot{\cup} \Gamma_C, \tag{4.18}$$

$$\sigma_P(\Lambda) = \begin{cases} \Gamma_P & \text{if } \gamma \neq -1, \quad (4.19) \\ \Gamma_P \dot{\cup} \{0\} & \text{if } \gamma = -1. \quad (4.20) \end{cases}$$

PROOF. Let us first show that $\Gamma_0 \subseteq \sigma_C(\Lambda)$. For $\lambda \in \Gamma_0$ we choose ξ in the complement of D_A and put $\bar{x} = 0$ and $\bar{y}(t) = e^{\lambda t}(\mu - A)^{-1}\xi$ where $\mu \in \rho(A)$ and $t \in [-r, 0]$. We observe that (3.6) becomes

$$\lambda x = \gamma r e^{-\lambda r} A(\mu - A)^{-1}\xi,$$

and this equation cannot have a solution x in D_A. By virtue of Proposition 3.3 it follows that $(\lambda - \Lambda)D_\Lambda \neq Z$ and therefore $\lambda \notin \rho(\Lambda)$. It is also obvious that $\lambda \notin \sigma_P(\Delta) = \sigma_P(\Lambda)$. To prove that $\lambda \notin \sigma_R(\Lambda)$ (and hence that $\lambda \in \sigma_C(\Lambda)$) it is sufficient now to show that $(\lambda - \Lambda)D_\Lambda$ is dense in Z. For this purpose we use Proposition 3.13 with $F_1 = D_A$ and $L_1 = L^2(-r, 0; D_{A^2})$. Since $F_1 + G_\lambda(L_1) \subseteq D_A = \Delta(\lambda)D_A$ it is easy to check (3.34) and $(3.35)_\lambda$. Now (3.17), (4.9) and (4.10) give

$$\Gamma \subseteq \rho(\Lambda) \subseteq \Gamma \dot{\cup} \Gamma_C \dot{\cup} \Gamma_0,$$

from (3.18) and (4.10) we obtain

$$\sigma_C(\Lambda) \subseteq \Gamma_C \dot{\cup} \Gamma_0,$$

and (3.19), (4.10) and (4.11) imply

$$\Gamma_R \subseteq \sigma_R(\Lambda) \subseteq \Gamma_R \dot{\cup} \Gamma_C \dot{\cup} \Gamma_0.$$

But $\Gamma_0 \subseteq \sigma_C(\Lambda)$ has been demonstrated earlier and hence (4.16)–(4.18) are verified. Finally (4.19)–(4.20) follow from (3.20), (4.12) and (4.13).

We can now establish the instability of the trivial solution of (4.2)–(4.3) for $|\gamma| > 1$.

THEOREM 4.3. *Let $|\gamma| > 1$. For any $\delta > 0$ there exists $z = (x, y) \in Z$, such that $\|z\|_Z \leqq \delta$ and for the solution u of* (4.2)–(4.3) *with initial datum z we have*

$$\limsup_{t \to \infty} \|u(t)\|_F = \infty. \tag{4.21}$$

PROOF. From (4.7), (4.8) and (4.17) it follows that there exists $\lambda \in \sigma(\Lambda)$ with $\operatorname{Re} \lambda > 0$. Hence, given $\delta > 0$, there exists $z \in Z$ with $\|z\|_Z \leqq \delta$ such that

$$\sup_{t>0} \| e^{\Lambda t} z \|_Z = \infty.$$

From (2.10) and (2.12) we have

(4.22) $$\limsup_{t\to\infty} (\| u(t)\|_F + \| u \|_{L^2(t-r,t;D_A)}) = +\infty.$$

From (4.2) we obtain for $t \geqq r$

(4.23) $$\| u' \|_{L^2(t-r,t;H)} \leqq \| Au \|_{L^2(t-r,t;H)} + |\gamma| \, \| Au \|_{L^2(t-2r,t-r;H)}.$$

Therefore if

$$\limsup_{t\to\infty} \| u \|_{L^2(t-r,t;D_A)} < \infty,$$

then

$$\limsup_{t\to\infty} \| u' \|_{L^2(t-r,t;H)} < \infty$$

and consequently from Theorem 3.1 in ([14], p. 23) we conclude that

$$\limsup_{t\to\infty} \| u(t) \|_F < \infty.$$

This proves that (4.22) is equivalent to (4.21).

In the case $|\gamma| \leqq 1$ under further conditions on A every solution of (4.1) is bounded on $[0,\infty)$, as will be proved at the end of this section. One is tempted to try to verify such a result by showing that

(4.24) $$s(\Lambda) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\Lambda)\}$$

is negative and that the type $\omega(\Lambda)$ of the semigroup $e^{\Lambda t}$ defined by

$$\omega(\Lambda) = \inf\{\omega \in R : \text{there exists } M > 0 \text{ with } \| e^{\Lambda t} \|_{\mathscr{L}(Z)} \leqq Me^{\omega t}\}$$

satisfies

(4.25) $$s(\Lambda) = \omega(\Lambda).$$

To ensure that (4.25) holds various conditions are known (see e.g. [19]). Those which seem to be easier to verify are: the compactness of $e^{\Lambda t}$ for some t or the ultimate differentiability of $e^{\Lambda t}$ (i.e. the differentiability of $t \to e^{\Lambda t}$ on $[\alpha,\infty)$ for some $\alpha > 0$). But in our case these properties do not hold. In fact we have the following result:

THEOREM 4.4. *Let $e^{\Lambda t}$ be the solution semigroup associated with* (4.2)–(4.3). *Then $e^{\Lambda t}$ is not ultimately differentiable. Moreover there is no $t_0 > 0$ such that $e^{\Lambda t_0}$ is compact.*

PROOF. From (4.17) we deduce that $\sigma(\Lambda)$ contains an unbounded subset Γ_0 lying on a vertical line. From Pazy's theorem (see [16]) Λ cannot generate a differentiable semigroup. In addition, $\Gamma_0 \subseteq \sigma_C(\Lambda)$ proves that $\sigma(\Lambda) \neq \sigma_P(\Lambda)$ and therefore there exists no t_0 such that $e^{\Lambda t_0}$ is compact (see [6], Theorem 2.20).

After this negative result we study the asymptotic behaviour of the solutions of (4.1)–(4.2) when $|\gamma| \leqq 1$ with a different method. We consider the particular but important case when A is selfadjoint.

THEOREM 4.5. *If $-A$ is positive and selfadjoint and $|\gamma| \leqq 1$, then each solution u of* (4.1)–(4.2) *satisfies*

$$\sup_{t>0} (|u(t)|_F + |u|_{L^2(t-r,t;D_A)}) < \infty. \tag{4.26}$$

PROOF. In view of (2.10) and (2.12) we need to show that

$$\sup_{t>0} \| e^{\Lambda t} \|_{\mathcal{L}(Z)} < \infty. \tag{4.27}$$

Let us define the scalar product

$$\langle z_1, z_2 \rangle_{Z_1} = \langle (-A)^{1/2} x_1, (-A)^{1/2} x_2 \rangle_H + \int_{-r}^{0} \langle Ay_1(s), Ay_2(s) \rangle_H ds \tag{4.28}$$

where $z_i = (x_i, y_i)$, $i = 1, 2$, and $\langle .\,,. \rangle_H$ denotes the scalar product in H. It is known that the domain of $(-A)^{1/2}$ endowed with the norm $\|(-A)^{1/2}x\|$ is equivalent to F (see e.g. [10], p. 665) and therefore $\sqrt{\langle z, z \rangle_{Z_1}}$ is a norm equivalent to $\|z\|$, as defined through (2.10). To prove (4.27) it is sufficient to show that for each $z = (y(0), y) \in D_\Lambda$ we have

$$\operatorname{Re}\langle \Lambda z, z \rangle_{Z_1} \leqq 0 \tag{4.29}$$

(see e.g. [9], Theorem 3.1.8). For $(y(0), y)$ we have the following estimate:

$$\begin{aligned} \operatorname{Re}\langle \Lambda z, z \rangle_{Z_1} &= \operatorname{Re}\langle (-A)^{1/2}[Ay(0) + \gamma Ay(-r)], (-A)^{1/2} y(0) \rangle_H \\ &\quad + \operatorname{Re} \int_{-r}^{0} \langle Ay'(s), Ay(s) \rangle_H ds \\ &= -\operatorname{Re}\langle Ay(0) + \gamma Ay(-r), Ay(0) \rangle + \frac{1}{2} \int_{-r}^{0} \frac{d}{ds} \| Ay(s) \|^2 ds \end{aligned}$$

$$
\begin{aligned}
&= -\tfrac{1}{2}\|Ay(0)\|^2 - \gamma \operatorname{Re}\langle Ay(-r), Ay(0)\rangle - \tfrac{1}{2}\|Ay(-r)\|^2 \\
&\leqq -\tfrac{1}{2}\|Ay(0)\|^2 + \|Ay(-r)\|\,\|Ay(0)\| - \tfrac{1}{2}\|Ay(-r)\|^2 \\
&= -\tfrac{1}{2}(\|Ay(0)\| - \|Ay(-r)\|)^2 \leqq 0,
\end{aligned}
$$

and the result is proved.

5. The distributed delay case

In this section we consider (2.5)–(2.6) with $A_1 = f = 0$, i.e.

$$\dot{u}(t) = Au(t) + \int_{-r}^{0} a(s)A_2u(t+s)ds, \qquad \text{for a.e. } t \geqq 0, \tag{5.1}$$

$$u(0) = x, \quad u(t) = y(t), \qquad \text{for a.e. } t \in [-r, 0]. \tag{5.2}$$

In contrast to the situation considered in the previous section, a regular weight function $a(.)$ is sufficient to ensure that the solution semigroup $e^{\mathcal{A}t}$ arising from (5.1), (5.2) is differentiable in $]r, \infty[$ and hence that (4.25) holds. (Later we will see that in general $e^{\mathcal{A}t}$ is not compact.)

THEOREM 5.1. *If $a(.) \in W^{1,2}(-r, 0)$, then $t \to e^{\mathcal{A}t}$ is differentiable in $]r, \infty[$.*

PROOF. Given $T > 0$ and $(x, y) \in Z$ let u be the solution of (5.1)–(5.2). As $u \in L^2(-r, T; D_A)$ and $a \in W^{1,2}(-r, 0)$ it can be seen that g, given by

$$g(t) = \int_{-r}^{0} a(s)A_2u(t+s)ds, \qquad \text{for } 0 \leqq t \leqq T, \tag{5.3}$$

belongs to $W^{1,2}(0, T; H)$. From (5.1) we get (see e.g. Theorem 2.3 of [8])

$$u(t) = e^{At}x + \int_0^t e^{A(t-s)}g(s)ds. \tag{5.4}$$

Writing

$$u(t) = e^{At}x + \int_0^t e^{As}g(0)ds + \int_0^t e^{A(t-s)}[g(s) - g(0)]ds \tag{5.5}$$

we can use (i) of Theorem 2.4 of [8] to prove that for each $T > 0$ the last term in (5.5) belongs to $W^{1,2}(0, T; D_A) \cap W^{2,2}(0, T; H) \subset C^1(0, T; F)$. From (5.5) it is easy to deduce that for each $\bar{t} > r$

$$u \in W^{1,2}(\bar{t} - r, \bar{t}; D_A) \tag{5.6}$$

and

(5.7) $$u' \in C(\bar{t}-r,\bar{t};F).$$

These properties show that (5.1) is satisfied for all $t \in [\bar{t}-r,\bar{t}]$.

In particular from (5.1) and (5.7) we obtain for $t=\bar{t}$

(5.8) $$Au(\bar{t})+\int_{-r}^{0} a(\theta)A_2u(\bar{t}+\theta)d\theta \in F.$$

If we recall the definition of $e^{\Lambda t}$ and the characterization of D_Λ given in (2.12) and (2.14) we recognize that (5.6) and (5.8) are equivalent to

(5.9) $$e^{\Lambda\bar{t}}z \in D_\Lambda.$$

As $\bar{t}$ is arbitrary in $]r,+\infty[$ we get the conclusion.

We pass now to the more particular case in which $A_2=A$ and consider

(5.10) $$\dot{u}(t)=Au(t)+\int_{-r}^{0} a(s)Au(t+s)ds, \qquad \text{for a.e. } t \geqq 0,$$

(5.11) $$u(0)=x; \quad u(t)=y(t), \qquad \text{for a.e. } t\in[-r,0].$$

From (3.1) we have for $\lambda \in \mathbf{C}$ and $x \in D_A$

(5.12) $$\Delta(\lambda)x = \lambda x - n(\lambda)Ax$$

where

(5.13) $$n(\lambda)=1+\int_{-r}^{0} a(s)e^{\lambda s}ds.$$

Analogous to Lemma 4.1 we obtain the following result:

LEMMA 5.1. *Let*

(5.14) $$\Gamma=\{\lambda\in\mathbf{C}: n(\lambda)\neq 0,\ \lambda n(\lambda)^{-1}\in\rho(A)\},$$

(5.15) $$\Gamma_C=\{\lambda\in\mathbf{C}: n(\lambda)\neq 0,\ \lambda n(\lambda)^{-1}\in\sigma_C(A)\},$$

(5.16) $$\Gamma_R=\{\lambda\in\mathbf{C}: n(\lambda)\neq 0,\ \lambda n(\lambda)^{-1}\in\sigma_R(A)\},$$

(5.17) $$\Gamma_P=\{\lambda\in\mathbf{C}: n(\lambda)\neq 0,\ \lambda n(\lambda)^{-1}\in\sigma_P(A)\},$$

(5.18) $$\Gamma_0=\{\lambda\in\mathbf{C}: \lambda\neq 0,\ n(\lambda)=0\},$$

then we have

(5.19) $$\rho(\Delta)=\Gamma,$$

(5.20) $$\sigma_C(\Delta)=\Gamma_C\,\dot{\cup}\,\Gamma_0,$$

$$\sigma_R(\Delta) = \Gamma_R, \tag{5.21}$$

$$\sigma_P(\Delta) = \begin{cases} \Gamma_P & \text{if } 1 + \int_{-r}^{0} a(s)ds \neq 0, & (5.22) \\ \Gamma_P \dot{\cup} \{0\} & \text{if } 1 + \int_{-r}^{0} a(s)ds = 0. & (5.23) \end{cases}$$

We can also derive the analogue to Proposition 4.2:

PROPOSITION 5.2.

$$\Gamma \subseteq \rho(\Lambda) \subseteq \Gamma \dot{\cup} \Gamma_C, \tag{5.24}$$

$$\Gamma_0 \subseteq \sigma_C(\Lambda) \subseteq \Gamma_0 \dot{\cup} \Gamma_C, \tag{5.25}$$

$$\Gamma_R \subseteq \sigma_R(\Lambda) \subseteq \Gamma_R \dot{\cup} \Gamma_C, \tag{5.26}$$

$$\sigma_P(\Lambda) = \begin{cases} \Gamma_P & \text{if } 1 + \int_{-r}^{0} a(s)ds \neq 0, & (5.27) \\ \Gamma_P \dot{\cup} \{0\} & \text{if } 1 + \int_{-r}^{0} a(s)ds = 0. & (5.28) \end{cases}$$

PROOF. Let us first show that $\Gamma_0 \not\subseteq \rho(\Lambda)$. If $\lambda \in \Gamma_0$, we have $\int_{-r}^{0} a(s)e^{\lambda s}ds = -1$ and hence there exists $\varphi \in C^1(-r,0)$ with $\varphi(0) = 0$, such that

$$\int_{-r}^{0} a(s)e^{\lambda s}\varphi(s)ds = c \neq 0. \tag{5.29}$$

If we choose ξ in the complement of D_A and set $\bar{x} = 0$ and

$$\bar{y}(s) = e^{\lambda s}\varphi'(s)(\mu - A)^{-1}\xi, \qquad \text{where } \mu \in \rho(A),$$

then (3.6) becomes

$$\lambda x = -cA(\mu - A)^{-1}\xi,$$

which cannot have a solution $x \in D_A$ by virtue of (5.29). From Proposition 3.3 we deduce that $(\lambda - \Lambda)D_\Lambda \neq Z$ and therefore $\lambda \notin \rho(\Lambda)$. Moreover $\Delta(\lambda)$ is injective, since $\lambda \neq 0$, and consequently $\lambda \notin \sigma_P(\Lambda)$ by virtue of Theorem 3.9. To prove that $\lambda \notin \sigma_R(\Lambda)$, we can use Proposition 3.13 with $F_1 = D_A$ and $L_1 = L^2(-r,0;D_{A^2})$, and conclude that $\Gamma_0 \subseteq \sigma_C(\Lambda)$. To obtain (5.24)–(5.28) we can now proceed exactly as in the proof of Proposition 4.2.

To derive more information on the sets Γ_0, Γ_C, Γ_R and Γ_P we now suppose that $a(s) \equiv a \in \mathbf{R} - \{0\}$, for $s \in [-r,0]$. In this case

$$n(\lambda) = 1 + a\int_{-r}^{0} e^{\lambda s}ds; \tag{5.30}$$

to stress the dependence of the sets defined in Lemma 5.1 on a, we write $\Gamma = \Gamma(a)$ and $\Gamma_i = \Gamma_i(a)$ with $i = C, R, P$ and 0. It is easy to see that $\lambda = x + iy \in \Gamma_0(a)$ if and only if

$$(x, y) \neq (0,0),$$

$$x + a(1 - e^{-rx}\cos ry) = 0, \tag{5.31}$$

$$y + ae^{-rx}\sin ry = 0. \tag{5.32}$$

THEOREM 5.3. *If $a(s) \equiv a \neq 0$ in $[-r, 0]$, then $\Gamma_0(a)$ is not empty and hence $e^{\Lambda t}$ is not compact for every $t > 0$.*

PROOF. If $a > 0$, then there exists $\lambda = x + iy \in \Gamma_0(a)$ with $y \in]\pi/r, 2\pi/r[$. In fact from (5.32) we obtain

$$x = \frac{1}{r}\log\left(-\frac{a\sin yr}{y}\right),$$

and substituting into (5.31) we have

$$\frac{y}{a}\cot ry + \frac{1}{ar}\log\left(-\frac{a\sin ry}{y}\right) + 1 = 0,$$

which has a solution in $]\pi/r, 2\pi/r[$. If $a < 0$ it suffices to consider y in $]2\pi/r, 3\pi/r[$. From (5.25) we deduce that $\sigma_C(\Lambda)$ is not empty and therefore $e^{\Lambda t}$ cannot be compact by virtue of Theorem 2.20 of [6].

We prove next a series of technical results which will allow us to locate the spectrum of Λ and consequently to study the asymptotic behaviour of (5.10), (5.11).

PROPOSITION 5.4. *There exists $\lambda = x + iy \in \Gamma_0(a)$ with $y = 0$ if and only if $a < 0$ and $a \neq -1/r$. If $a < -1/r$ we have $x > 0$ and if $-1/r < a < 0$ we have $x < 0$.*

PROOF. The result is a consequence of the study of the roots of $x + a(1 - e^{-rx}) = 0$.

PROPOSITION 5.5. *If there exists $\lambda = x + iy \in \Gamma_0(a)$ and $a \geqq -1/r$, then $x < 0$.*

PROOF. By virtue of the preceding proposition we can suppose that $y \neq 0$. If $-1/r \leqq a < 0$ we derive from (5.32) that

$$are^{-rx}\frac{\sin ry}{ry} = -1 \leqq ar \quad \text{and hence} \quad e^{-rx}\frac{\sin ry}{ry} \geqq 1.$$

We conclude that

$$1 > \frac{\sin ry}{ry} \geqq e^{rx}$$

and therefore $x < 0$.

In the next propositions we investigate the location of the set $\Gamma_R(a)$ and $\Gamma_P(a)$, when $\sigma_R(A)$ and $\sigma_P(A)$ contain negative real numbers. More precisely, for each $\varepsilon > 0$ and $a \in R$ we consider the set

$$\gamma(\varepsilon, a) = \{\lambda \in \mathbf{C}\colon n(\lambda) \neq 0,\ \lambda n(\lambda)^{-1} = -\varepsilon\} \tag{5.33}$$

where $n(\lambda)$ is defined by (5.30). It can easily be seen that $\lambda \in \gamma(\varepsilon, a)$ if and only if

$$\begin{aligned} &\lambda \neq 0, \\ &f(\lambda, a) = \lambda^2 + \varepsilon\lambda + \varepsilon a(1 - e^{-\lambda r}) = 0, \end{aligned} \tag{5.34}$$

or, setting $\lambda = x + iy$ and $f = \varphi + i\psi$,

$$\begin{aligned} &(x, y) \neq (0,0), \\ &\varphi(x, y, a) = x^2 - y^2 + \varepsilon x + \varepsilon a(1 - e^{-xr} \cos yr) = 0, \\ &\psi(x, y, a) = 2xy + \varepsilon y + \varepsilon a\, e^{-xr} \sin yr = 0. \end{aligned} \tag{5.35}$$

PROPOSITION 5.6. *If $\lambda = x + iy \in \gamma(\varepsilon, a)$ and $|a| \leqq 1/r$, then $x < 0$.*

PROOF. Suppose that $\lambda = x + iy$ with $x \geqq 0$ satisfies $\lambda = -\varepsilon n(\lambda)$. Taking the real parts of both sides we get with (5.30)

$$x = -\varepsilon\left(1 + a \int_{-r}^{0} e^{xs} \cos ys\,ds\right),$$

hence

$$-a \int_{-r}^{0} e^{xs} \cos ys\,ds = 1 + x/\varepsilon \geqq 1,$$

and therefore

$$|a| \geqq \left|\int_{-r}^{0} e^{xs} \cos ys\,ds\right|^{-1} > \left(\int_{-r}^{0} e^{xs}\,ds\right)^{-1} \geqq r^{-1},$$

since $x \geqq 0$, which is the desired contradiction.

It will be useful to consider (5.35) as a system implicitly defining two functions $x(a)$ and $y(a)$. Let us introduce the Jacobian

$$J(x,y,a) = \begin{vmatrix} \varphi_x(x,y,a) & \varphi_y(x,y,a) \\ \psi_x(x,y,a) & \psi_y(x,y,a) \end{vmatrix} \tag{5.36}$$

and the set

$$K = \{(x,y,a)\colon \varphi(x,y,a) = \psi(x,y,a) = 0,\ J(x,y,a) \neq 0\}. \tag{5.37}$$

From the implicit function theorem (IFT) it is known that if $(\bar{x}, \bar{y}, \bar{a}) \in K$, then there exists $\delta > 0$ and two differentiable functions $x : [\bar{a} - \delta, \bar{a} + \delta] \to R$ and $y : [\bar{a} - \delta, \bar{a} + \delta] \to R$ such that

$$\{(x(a), y(a), a)\colon |a - \bar{a}| \leqq \delta\}$$
$$= \{(x,y,a)\colon \varphi(x,y,a) = \psi(x,a,y) = 0,\ |a - \bar{a}| \leqq \delta,\ |y - \bar{y}| \leqq \delta,\ |x - \bar{x}| \leqq \delta\},$$

and

$$x'(a) = -J(x,y,a)^{-1} \begin{vmatrix} \varphi_a(x,y,a) & \varphi_y(x,y,a) \\ \psi_a(x,y,a) & \psi_y(x,y,a) \end{vmatrix}. \tag{5.38}$$

PROPOSITION 5.7. *Let $a > -1/r$. If (x,y,a) satisfies* (5.35) *and $J(x,y,a) = 0$, then $x < 0$.*

PROOF. We calculate

$$J(x,y,a) = (2x + \varepsilon(1 + are^{-xr}\cos yr))^2 + (2y - \varepsilon are^{-xr}\sin yr)^2, \tag{5.39}$$

and therefore $J(x,y,a) = 0$ if and only if $\lambda = x + iy$ satisfies

$$2\lambda + \varepsilon(1 + are^{-r\lambda}) = 0. \tag{5.40}$$

Moreover $\varphi(x,y,a) = \psi(x,y,a) = 0$ is (compare (5.34)) equivalent to

$$\lambda^2 + \varepsilon\lambda + \varepsilon a(1 - e^{-\lambda r}) = 0. \tag{5.41}$$

From (5.40), (5.41) we deduce

$$r\lambda^2 + (\varepsilon r + 2)\lambda + \varepsilon(ar + 1) = 0.$$

As $a > -1/r$ this last equation implies $\operatorname{Re}\lambda < 0$ and the claim is verified.

PROPOSITION 5.8. *The triplet $(a, 0, y)$ is a solution of* (5.35) *if and only if $y = \pm y_k$, $a = a_k = -y_k/\sin ry_k$, $k \in \mathbf{N}$, where $y_1 < y_2 < \cdots$ are the positive* *solutions of*

$$y + \varepsilon \tan \frac{ry}{2} = 0. \tag{5.42}$$

Moreover $a_k > 1/r$ *for each* $k \in \mathbf{N}$.

We note that y_k and a_k depend on ε.

PROOF. Setting $x = 0$ in (5.35) we obtain $y \neq 0$ and

$$\begin{aligned} &-y^2 + \varepsilon a(1 - \cos yr) = 0, \\ &y + a \sin yr = 0, \end{aligned} \tag{5.43}$$

and hence

$$\begin{aligned} &y^2 = 2\varepsilon a \sin^2 \frac{yr}{2}, \\ &a = \frac{-y}{\sin yr}. \end{aligned} \tag{5.44}$$

From the last equality it follows that $a > 1/r$. Moreover

$$-y = 2\varepsilon \frac{\sin^2 \frac{yr}{2}}{2 \sin \frac{yr}{2} \cos \frac{yr}{2}} = \varepsilon \tan \frac{yr}{2},$$

which is (5.42). This ends the proof.

PROPOSITION 5.9. *For each* $k \in \mathbf{N}$ *there exists a continuous function* $\lambda_k : I_k \to \mathbf{C}$ *such that*

(i) $I_k =]a_k - \delta_k, a_k + \delta_k[$ *with* $\delta_k > 0$ *and* a_k *given in Proposition* 5.8,
(ii) $f(\lambda_k(a), a) = 0$ *for* $a \in I_k$,
(iii) $\lambda_k(a_k) = y_k$,
(iv) $\operatorname{Re} \lambda_k(a) < 0$ *for* $a_k - \delta_k < a < a_k$,
(v) $\operatorname{Re} \lambda_k(a) > 0$ *for* $a_k < a < a_k + \delta_k$.

PROOF. As $(a_k, 0, y_k)$ satisfies (5.35) and $a_k > 1/r$ we know from Proposition 5.7 and (5.39) that

$$J(a_k, 0, y_k) > 0, \tag{5.45}$$

and therefore the IFT implies the existence of a function λ_k satisfying (i)–(iii). If $(a, 0, k) \in K$ we get from (5.38) and (5.39)

$$x'(a) = -J^{-1}(a,0,y)\begin{vmatrix} \varepsilon(1-\cos yr) & -2y+\varepsilon ar\sin yr \\ \varepsilon \sin yr & \varepsilon(1+ar\cos yr)\end{vmatrix}. \tag{5.46}$$

If $a = a_k$ and $y = y_k$ we can use (5.42) and (5.43) to obtain

$$x'(a_k) = -y_k^2\varepsilon a_k^{-1}J(a_k,0,y_k)^{-1}(a_k r\cos y_k r - 1 - \varepsilon r). \tag{5.47}$$

From (5.43) and (5.42) it follows that

$$a_k r\cos y_k r = -ry_k\cot y_k r = \frac{\varepsilon r}{2}\left(1-\tan^2\frac{y_k r}{2}\right),$$

and therefore

$$x'(a_k) = y_k^2\varepsilon a_k^{-1}J(a_k,0,y_k)^{-1}\left(1+\frac{\varepsilon r}{2}+\frac{\varepsilon r}{2}\tan^2\frac{y_k r}{2}\right).$$

Using (5.45) we deduce that $x'(a_k) > 0$ and (iv)–(v) easily follow.

REMARK. The same result holds if we replace y_k with $-y_k$ in (iii).

From Propositions 5.8 and 5.9 we deduce the existence of elements $\lambda \in \gamma(\varepsilon, a)$, as a varies in a neighborhood of a_k; to determine for which values of a there exist $\lambda \in \gamma(\varepsilon, a)$ with $\operatorname{Re}\lambda > 0$, we use the following

PROPOSITION 5.10. *Let* (λ_0, a_0) *with* $\operatorname{Re}\lambda_0 > 0$ *and* $a_0 > 1/r$ *satisfy*

$$f(\lambda_0, a_0) = 0. \tag{5.48}$$

Then there exist a' *and a function* $\lambda :]a', \infty[\to \mathbf{C}$ *such that* $1/r \leqq a' < a_0$ *and*

$$\begin{aligned} &\lambda(a_0) = \lambda_0, \\ &f(\lambda(a), a) = 0 \qquad \text{for } a > a', \\ &\operatorname{Re}\lambda(a) > 0 \qquad \text{for } a > a'. \end{aligned} \tag{5.49}$$

Moreover there exists $y' \in \mathbf{R}$ *such that*

$$f(iy', a') = 0. \tag{5.50}$$

PROOF. Let us consider the family $\mathcal{F}$ of continuous functions

$$\lambda : I_\lambda \to \mathbf{C} \tag{5.51}$$

where I_λ is an open interval containing a_0 and such that

$$\lambda(a_0)=\lambda_0,$$
$$\text{(5.52)}\qquad \operatorname{Re}\lambda(a)>0 \qquad \text{for } a\in I_\lambda,$$
$$f(\lambda(a),a)=0 \qquad \text{for } a\in I_\lambda.$$

By Proposition 5.7 and the IFT we see that $\mathcal{F}$ is not empty and from Proposition 5.6 we deduce that $I_\lambda\subseteq]1/r,+\infty[$. Let us prove that if λ_1 and λ_2 belong to $\mathcal{F}$ then $\lambda_1=\lambda_2$ in $I_{\lambda_1}\cap I_{\lambda_2}$. In fact by the IFT there exists an interval containing a_0 where $\lambda_1=\lambda_2$ and so we can deduce the existence of a maximal interval J such that $a_0\in J\subseteq I_{\lambda_1}\cap I_{\lambda_2}$ and $\lambda_1=\lambda_2$ in J. If J is different from $I_{\lambda_1}\cap I_{\lambda_2}$ then one of its endpoints a^* must belong to $I_{\lambda_1}\cap I_{\lambda_2}$. By continuity we get $\lambda_1(a^*)=\lambda_2(a^*)$ and since $\operatorname{Re}\lambda_1(a^*)>0$, Proposition 5.7 and the IFT imply the desired contradiction. Having established this fact the following definition of the function λ^* is justified:

$$\lambda^*: I^*\to \mathbf{C},$$
$$\text{(5.53)}\qquad I^*=\bigcup_{\lambda\in\mathcal{F}} I_\lambda,$$
$$\lambda^*(a)=\lambda(a) \qquad \text{if } a\in I_\lambda \text{ with } \lambda\in\mathcal{F}.$$

We set $I^*=]a',a''[$. Let $\{a'_n\}\subseteq I^*$ be a sequence such that

$$\text{(5.54)}\qquad \lim_{n\to\infty} a'_n=a'.$$

Set $x(a)=\operatorname{Re}\lambda^*(a)$ and $y(a)=\operatorname{Im}\lambda^*(a)$. If $y(a'_n)=0$ we get by (5.35)

$$0\leqq x^2(a'_n)=-\varepsilon x(a'_n)-\varepsilon a'_n+\varepsilon a'_n e^{-rx(a'_n)}\leqq \varepsilon a'_n$$

and if $y(a'_n)\neq 0$,

$$\text{(5.55)}\qquad 0\leqq 2x(a'_n)=-\varepsilon-\varepsilon a'_n r e^{-rx(a'_n)}\frac{\sin ry(a'_n)}{ry(a'_n)}\leqq \varepsilon a'_n r.$$

Moreover

$$\text{(5.56)}\qquad y^2(a'_n)=x^2(a'_n)+\varepsilon[x(a'_n)+a'_n]-\varepsilon a'_n e^{-rx(a'_n)}\cos ry(a'_n)$$

and so $\{x(a'_n)\}$ and $\{y(a'_n)\}$ are bounded; by possibly choosing a subsequence we can suppose that there exist

$$\text{(5.57)}\qquad \lim_{n\to\infty} x(a'_n)=x'\geqq 0$$

and

$$\lim_{n\to\infty} y(a'_n) = y'. \tag{5.58}$$

From (5.54), (5.57) and (5.58) we deduce

$$f(x' + iy', a') = 0.$$

Now if $x' > 0$ Proposition 5.7 and the IFT imply the existence of a function belonging to $\mathcal{F}$ defined on an interval which is not contained in I^*. This is a contradiction and so (5.50) holds. Finally if $a'' < \infty$ with an analogous argument we can show that there exists $a''_n < a''$ such that $\lim_{n\to\infty} a''_n = a''$, $\lim_{n\to\infty} x(a''_n) = 0$, $\lim_{n\to\infty} y(a''_n) = y''$ and $x(a''_n) > 0$. This implies $f(iy'', a'') = 0$ and so $y'' = \pm y_k$, $a'' = a_k$ for suitable $k \in \mathbf{N}$, by virtue of Proposition 5.8. By using the remark after Proposition 5.9 and IFT we see that $\lim_{n\to\infty} a''_n = a_k$ and $x(a''_n) > 0$ are in contradiction with (iv) of Proposition 5.9 and so we must have $a'' = +\infty$. Properties (5.49) are thus verified by the function $\lambda = \lambda^*$ and the result is proved.

PROPOSITION 5.11.

(i) *If* $\lambda_0 \in \gamma(\varepsilon, a_0)$ *with* $1/r < a_0 < a_1$ (*where* a_1 *is defined in Proposition* 5.8) *then* $\operatorname{Re} \lambda_0 < 0$.

(ii) *If* $\lambda_0 \in \gamma(\varepsilon, a_1)$ *and* $\lambda_0 \neq \pm iy_1$ *then* $\operatorname{Re} \lambda_0 < 0$.

(iii) *If* $a_0 > a_1$, *then there exists* $\lambda_0 \in \gamma(\varepsilon, a_0)$ *with* $\operatorname{Re} \lambda_0 > 0$.

PROOF. Let $\lambda_0 \in \gamma(\varepsilon, a_0)$ with $1/r < a_0 < a_1$. From Proposition 5.8 we know that $\operatorname{Re} \lambda_0 \neq 0$ and the same proposition and Proposition 5.10 imply that if $\operatorname{Re} \lambda_0 > 0$ then there exists $a_k < a_0$ for suitable $k \in \mathbf{N}$, which is absurd, and therefore $\operatorname{Re} \lambda_0 < 0$ which proves (i). The proof of (ii) is obtained in an analogous way. Finally from Proposition 5.9 we have that given $\bar{a}_0 \in \,]a_1, a_1 + \delta[$ there exists $\bar{\lambda}_0 \in \gamma(\varepsilon, \bar{a}_0)$ with $\operatorname{Re} \bar{\lambda}_0 > 0$; consequently we can apply Proposition 5.10 and obtain (iii).

We are now in position to determine the sign of $\operatorname{Re} \lambda$ where λ belongs to $\Gamma_0(a)$ or to $\gamma(\varepsilon, a)$ as a varies in $(-\infty, \infty)$.

PROPOSITION 5.12. *For fixed* $\varepsilon > 0$, *let* $a_1 = a_1(\varepsilon)$ *and* $y_1 = y_1(\varepsilon)$ *be defined in Proposition* 5.8.

(i) *If* $a < -1/r$, *then there exist elements* $\lambda \in \Gamma_0(a)$ *with* $\operatorname{Re} \lambda > 0$.

(ii) *If* $-1/r \leqq a < a_1$, $a \neq 0$ *and* $\lambda \in \Gamma_0(a) \cup \gamma(\varepsilon, a)$, *then* $\operatorname{Re} \lambda < 0$.

(iii) *If* $a = a_1$, *then* $\pm iy_1 \in \gamma(\varepsilon, a)$ *and if* $\lambda \in \Gamma_0(a) \cup \gamma(\varepsilon, a)$ *with* $\lambda \neq iy_1$, *then* $\operatorname{Re} \lambda < 0$.

(iv) *If* $a > a_1$ *then there exist elements* $\lambda \in \gamma(\varepsilon, a)$ *with* $\operatorname{Re} \lambda > 0$.

PROOF. Proposition 5.4 implies (i), whereas (ii) follows from Propositions 5.5, 5.6 and 5.11 (i). We also observe that Propositions 5.5, 5.8 and 5.11 (ii) imply (iii). Finally (iv) is (iii) of the preceding proposition.

With the results proved up to now we can describe the asymptotic behaviour of the solutions of (5.1)–(5.2) when $a(.)$ is a constant different from zero and $A_2 = A$.

THEOREM 5.13. *Let $a \in \mathbf{R} - \{0\}$, $z = (x, y) \in Z$ and let u denote the solution of*

$$\dot{u}(t) = Au(t) + a \int_{-r}^{0} Au(t+s)ds \qquad \text{for a.e. } t \geqq 0, \tag{5.59}$$

$$u(0) = x, \quad u(t) = y(t) \qquad \text{for a.e. } t \in [-r, 0], \tag{5.60}$$

given by Theorem 2.1.

For each $\varepsilon > 0$ let $y_1(\varepsilon)$ be the smallest positive solution of $y + \varepsilon \tan(ry/2) = 0$ and set

$$a_1(\varepsilon) = \frac{-y_1(\varepsilon)}{\sin(ry_1(\varepsilon))}$$

(see Proposition 5.8).

If

$$a < -1/r \tag{5.61}$$

or

$$a > \inf\{a_1(\varepsilon) : \varepsilon > 0, -\varepsilon \in \sigma_R(A) \cup \sigma_P(A)\} \tag{5.62}$$

(when $\{\varepsilon > 0 : -\varepsilon \in \sigma_R(A) \cup \sigma_P(A)\}$ is not empty), then given any $\delta > 0$ there exists $z \in Z$ with $\|z\|_Z \leqq \delta$ and

$$\limsup_{t \to \infty} \|u(t)\|_F = +\infty. \tag{5.63}$$

If

$$a = -1/r \tag{5.64}$$

or

$$a = a_1(\varepsilon) \qquad \text{for a suitable } \varepsilon > 0 \text{ with } -\varepsilon \in \sigma_P(A), \tag{5.65}$$

then there exists $z \neq 0$ in Z such that

$$\|u(t)\|_Z = \|z\|_Z \qquad \text{for all } t \geqq 0. \tag{5.66}$$

Finally if

(5.67) $\sigma(A) \subseteq]-\infty, -\bar{\varepsilon}]$ *with* $\bar{\varepsilon} > 0$ *and* $a \in]-1/r, \inf\{a_1(\varepsilon): -\varepsilon \in \sigma(A)\}[$,

then there exist M, $\omega > 0$ *such that for every* $z \in Z$ *we have*

(5.68) $$\| u(t) \|_F + \| u \|_{L^2(t-r,t;D_A)} \leqq M e^{-\omega t} \| z \|_Z \qquad \text{for } t \geqq 0.$$

REMARK. If $\sigma_R(A) \cup \sigma_P(A) \subseteq]-\infty, -\bar{\varepsilon}]$, $\bar{\varepsilon} > 0$, then

$$\inf\{a_1(\varepsilon): \varepsilon \in \sigma_R(A) \cup \sigma_P(A)\} \geqq \frac{-y_1(\beta)}{\sin(r y_1(\beta))}$$

where β is the solution of $y = \tan ry$ in $[\pi/r, 3\pi/2r]$.

PROOF. If (5.61) holds, we deduce from Proposition 5.12 (i) and (5.25) that there exists $\lambda \in \Gamma_0(a) \subseteq \sigma_C(\Lambda)$ with $\operatorname{Re}\lambda > 0$. On the other hand, from (5.62) and (5.33), (5.16), (5.17) and (5.26)–(5.28) we obtain that for suitable $-\varepsilon' \in \sigma_R(A) \cup \sigma_P(A)$ we have $a_1(\varepsilon') < a$ and $\gamma(\bar{\varepsilon}, a) \subseteq \Gamma_R(a) \cup \Gamma_P(a) \subseteq \sigma(\Lambda)$. Therefore from Proposition 5.12 (iv) we again deduce that there exists some $\lambda \in \sigma(\Lambda)$ with $\operatorname{Re}\lambda > 0$. To prove (5.63) we argue as in Theorem 4.3. In our situation the equivalence of (4.21) and (4.22) is proved by using the following estimate which is obtained from (5.59) for each $t > r$:

(5.69) $$\| \dot{u} \|_{L^2(t-r,t;H)} \leqq \| Au \|_{L^2(t-r,t;H)} + ra\{\| Au \|_{L^2(t-2r,t-r;H)} + \| Au \|_{L^2(t-r,t;H)}\}.$$

To verify (5.66) we note that it suffices to argue the existence of some $\lambda \in \sigma_P(\Lambda)$ with $\operatorname{Re}\lambda = 0$. If $a = -1/r$, then $0 \in \sigma_P(\Lambda)$ by (5.28). On the other hand if (5.65) is satisfied, then by Proposition 5.12 (iii), (5.17), (5.27) and (5.33) we get $iy_1 \in \gamma(\bar{\varepsilon}, a_1(\bar{\varepsilon})) \subseteq \sigma_P(\Lambda)$. Let us prove the last part of the theorem. As $e^{\Lambda t}$ is differentiable (see Theorem 5.1), (4.25) holds and so it is sufficient to prove

(5.70) $$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\Lambda)\} < 0.$$

But from the characterization of the spectrum of the generator of a differentiable semigroup [16], we know that the part of $\sigma(\Lambda)$ contained in a vertical strip is a compact subset of **C** and hence (5.70) is equivalent to

(5.71) $$\lambda \in \sigma(\Lambda) \Rightarrow \operatorname{Re}\lambda < 0.$$

Therefore the proof is accomplished, because from (5.67) and (5.25)–(5.27) we get

$$\sigma(\Lambda) \subseteq \Gamma_0(a) \cup \Gamma_C(a) \cup \Gamma_R(a) \cup \Gamma_P(a) \subseteq \Gamma_0(a) \cup \Big(\bigcup_{\varepsilon > \bar{\varepsilon}} \gamma(\varepsilon, a) \Big).$$

Proposition 5.12 (ii) implies (5.71) and the proof is completed.

6. Example

In this section we give an application to the initial-boundary value problem of Dirichlet type for the retarded Laplace equation:

$$(6.1)\quad \begin{gathered} u_t(t,x) = \Delta u(t,x) + \gamma \Delta u(t-r,x) + a \int_{-r}^{0} \Delta u(t+s,x)\,ds + f(t,x) \\ \text{for a.e. } (t,x) \in [0,T] \times \Omega, \end{gathered}$$

$$(6.2)\quad \begin{gathered} u(t,x) = u_0(t,x) \qquad \text{for a.e. } (t,x) \in [-r,0] \times \Omega; \\ u(0,x) = \varphi(x) \qquad \text{for a.e. } x \in \Omega, \end{gathered}$$

$$(6.3)\quad u(t,x) = 0 \qquad \text{for a.e. } (t,x) \in [0,T] \times \partial\Omega.$$

Here Ω is a bounded open subset of R^n with "regular" boundary $\partial\Omega$; $\gamma, a \in \mathbf{R}$, $r > 0$, $T > 0$ and f, u_0 and φ are given functions.

We can write (6.1)–(6.3) as an initial boundary problem (2.5)–(2.6) in the Hilbert space $H = L^2(\Omega)$ by setting

$$(6.4)\quad \begin{aligned} A &= \Delta, \\ D_A &= \tilde{W}_0^{2,2}(\Omega) \doteq W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \\ A_1 &= \gamma\Delta, \qquad A_2 = a\Delta. \end{aligned}$$

It can be proved that F is equivalent to $W_0^{1,2}(\Omega)$ (see section 5 of [8]). For each $f \in L^2([0,T] \times \Omega), u_0 \in L^2(-r,0;\tilde{W}_0^{2,2}(\Omega))$ and $\varphi \in W_0^{1,2}(\Omega)$ there exists a unique

$$u \in L^2(-r,T;\tilde{W}_0^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \cap C(0,T;W_0^{1,2}(\Omega))$$

satisfying (6.1)–(6.3) as a consequence of Theorem 2.1. If $f = 0$ we obtain a solution defined in $[0, +\infty[$ and we can also apply the results of Sections 4 and 5 to study the asymptotic behaviour of the solutions of (6.1)–(6.3). Note that A is a selfadjoint and negative operator and its spectrum satisfies $\sigma(A) = \sigma_P(A) \subseteq]-\infty, -\bar{\varepsilon}]$ with $\bar{\varepsilon} > 0$.

THEOREM 6.1. *Let u be the solution of* (6.1)–(6.3) *when* $f = 0$.

(i) *If $a = 0$ and $|\gamma| > 1$, then for every $\delta > 0$ there exist an initial datum (u_0, φ) with*

$$(6.5)\quad \left(\int_{-r}^{0} \|u_0(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2} + \|\varphi\|_{W^{1,2}(\Omega)} < \delta$$

such that u satisfies

$$\limsup_{t\to\infty} \|u(t,.)\|_{W^{1,2}(\Omega)} = +\infty. \tag{6.6}$$

(ii) *If* $a=0$ *and* $|\gamma|\leqq 1$, *then for every solution we have*

$$\sup_{t>0}\left\{\|u(t,.)\|_{W^{1,2}(\Omega)} + \left(\int_{t-r}^{t} \|u(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2}\right\} < \infty. \tag{6.7}$$

(iii) *If* $\gamma=0$ *and* $a\notin[-1/r, \inf\{a_1(\varepsilon), -\varepsilon\in\sigma(A)\}]$ (*where* $a_1(\varepsilon)$ *is given in Theorem* 5.13) *then for every* $\delta>0$ *there exist* (u_0,φ) *satisfying* (6.5) *such that u satisfies* (6.6).

(iv) *If* $\gamma=0$ *and* $a=-1/r$ *or* $a=a_1(\varepsilon)$ *with* $-\varepsilon\in\sigma(A)$, *then there exists* $(u_0,\varphi)\neq(0,0)$ *such that*

$$\|u(t,.)\|_{W^{1,2}(\Omega)} + \left(\int_{t-r}^{t} \|u(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2} = \left(\int_{-r}^{0} \|u_0(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2} + \|\varphi\|_{W^{1,2}(\Omega)}. \tag{6.8}$$

(v) *If* $\gamma=0$ *and* $a\in\,]-1/r, \inf\{a_1(\varepsilon), -\varepsilon\in\sigma(A)\}[$ *then there exist M and* $\omega>0$ *such that for every solution u we have*

$$\begin{aligned}&\|u(t,.)\|_{W^{1,2}(\Omega)} + \left(\int_{t-r}^{t} \|u(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2}\\ &\leqq Me^{-\omega t}\left\{\left(\int_{-r}^{0} \|u_0(s,.)\|^2_{W^{2,2}(\Omega)}ds\right)^{1/2} + \|\varphi\|_{W^{1,2}(\Omega)}\right\}.\end{aligned} \tag{6.9}$$

Proof. The assumptions on A required by Theorems 4.3, 4.5 and 5.13 are satisfied and their conclusions give (i), (ii) and (iii)–(v) respectively.

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